LHS BOARD (Summary)

Operations on functions

: $f+g$, $f-g$, $f\cdot g$, $f\div g$

and

: Composition $f\circ g$

Composition

① : the function $f(x)=x$ is the identity element in the set of functions under the operation of composition. Sometimes we write $I(x)=x$, "I" for "identity".

① : Note that

$I\circ g = g\circ I$.

While composition is not in general commutative, composition with I is commutative.

RHS BOARD

[Ex 1] $g(x) = \dfrac{x^5+3x^2-x^{\frac{1}{3}}}{15}$

$f(x) = x$

a) $(g\circ f)(x) = g(f(x))$

Since $f(x)=x$,

$g(f(x)) = g(x)$

$= \dfrac{x^5+3x^2-x^{1/3}}{15}$

Conclusion. If $f: f(x)=x$ then $(g\circ f)(x) = g(x)$

b) $(f\circ g)(x) = f(g(x))$,

Since $f(x)=x$,

$f(g(x)) = g(x)$

$= \dfrac{x^5+3x^2-x^{1/3}}{15}$

Conclusion. If $f: f(x)=x$, then $(f\circ g)(x) = g(x)$.

① In general, $f: f(x)=x$ and g is any function,
$(f\circ g)(x) = f(g(x)) = g(x)$, and
$(g\circ f)(x) = g(f(x)) = g(x)$.

What role is played by the function $f(x)=x$?

LHS BOARD

Thm: g^{-1}, the inverse of g, exists iff g is a 1-1 function.

RHS BOARD

Once we have an identity element, what is the next question to raise?

Q. Do inverse elements exist for any or all functions?

E.g. Each number posses an inverse under $\bullet$,

$$2 \cdot \frac{1}{2} = 1$$

$$a \cdot \frac{1}{a} = 1 \quad , \ a \neq 0$$

and for addition,

$$2 + (-2) = 0$$

$$a + (-a) = 0 .$$

ways to think about inverse

$\boxed{\text{BIG Q:}}$ what about for functions?

① What do we mean when we say that under, say multiplication, a number has an inverse

a is the inverse b if $a \cdot b = 1$

So, for functions,

f is the inverse of g if $(f \circ g)(x) = I(x) = x$

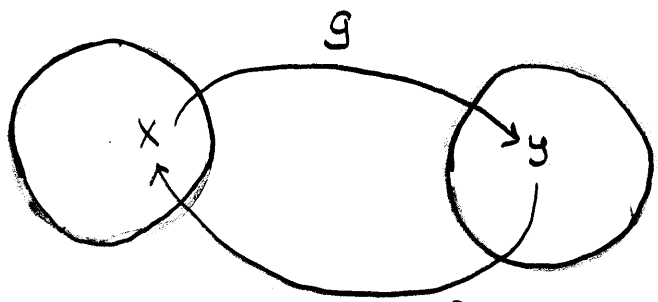
② an inverse function "undoes" the work of a function.

$$g(x) = x^3 \qquad (f \circ g)(x) = \sqrt[3]{x^3} = x$$

$$f(x) = \sqrt[3]{x^3}$$

right back where you started

③



Inverse of g
called g^{-1}

④

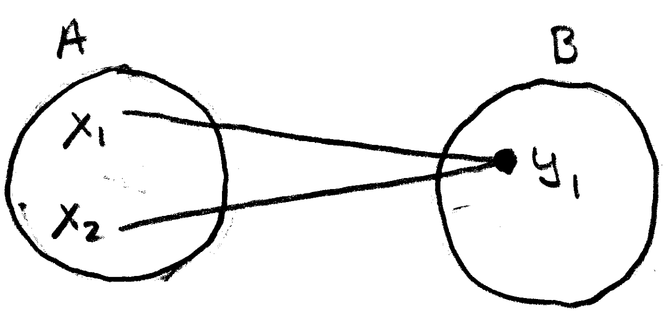
$x \longrightarrow \boxed{g} \longrightarrow \boxed{g^{-1}} \longrightarrow x$ ← just what you started with!

g^{-1} "undoes" g

Well, if g has an inverse function g^{-1}, what is the most basic requirement of g^{-1}?

<u>Ans</u>: g^{-1} must itself be a function (an element in the set of functions), just like $-a$ must be a number if $a+(-a)=0$).

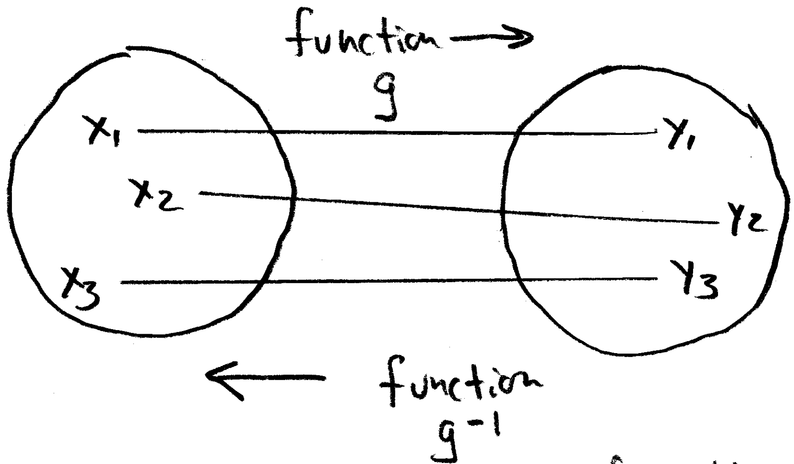
So, when is g^{-1} a function?



From A to B function

but from B to A, No function. WHY?

→



Now the way back is a function!

When a function has this quality, we say the function is a 1-1 function ("a one to one function")

NB: We will want a rigorous definition of this quality. I.e. a definition we can use to prove this quality belongs to a function

ON LHS BOARD:

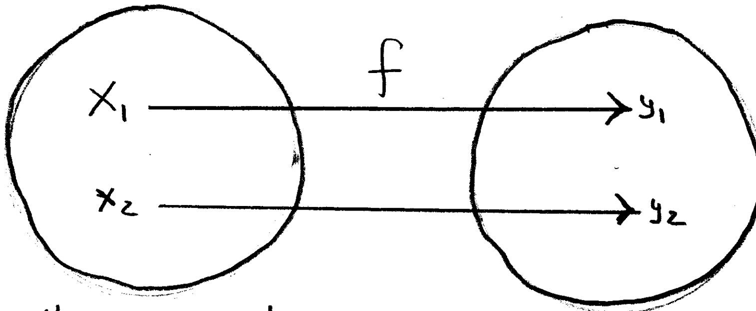
☆ g^{-1}, the inverse of a function g, exists iff g is a 1-1 function.

→

Useful definition of 1-1 function.

Def: f is a 1-1 function if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

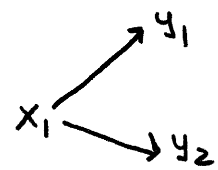


if these two are the same

then these two are the same.

in other words one y cannot go to different x's.

f is function: one x cannot go to two y



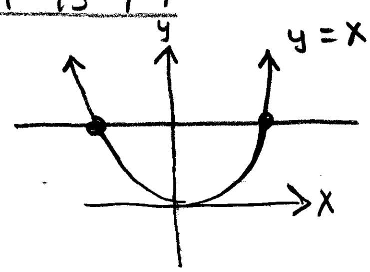
NOT Function

1-1 function: one y cannot go to two x.

this insures that the inverse of f is itself a function.

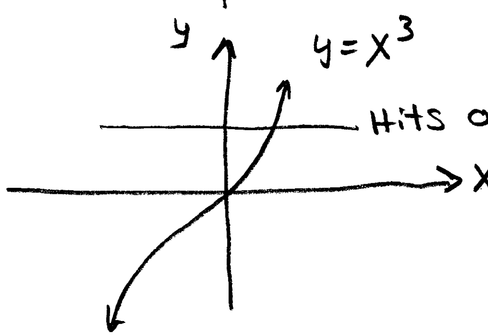
How to get an idea that f is 1-1

Horizontal Line test



$y = x^2$

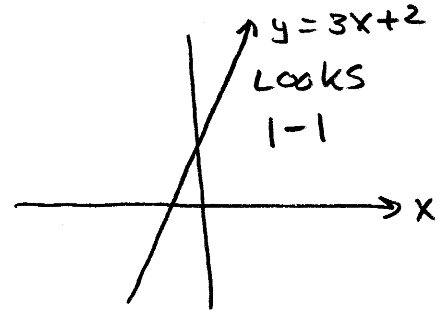
Hits Graph twice, So $y = x^2$ not 1-1



$y = x^3$

Hits once $\Rightarrow$ $y = x^3$ 1-1 function

How to prove f is 1-1



[EX2] Prove $f(x) = 3x + 2$ is 1-1.

Proof:

Spose $f(x_1) = f(x_2)$.
(Show x_1 must be x_2.)

$$f(x_1) = 3x_1 + 2$$
$$f(x_2) = 3x_2 + 2$$

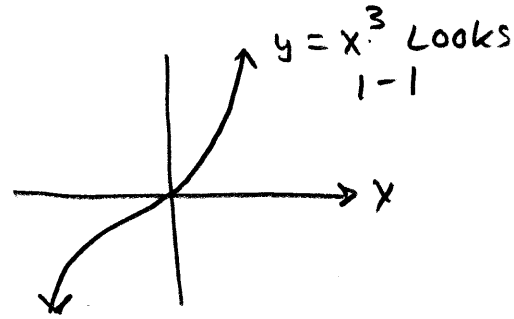
if $f(x_1) = f(x_2)$, then $3x_1 + 2 = 3x_2 + 2$

$$\Rightarrow 3x_1 = 3x_2$$
$$\Rightarrow x_1 = x_2$$

□

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $f(x) = 3x+2$ is 1-1.

★ ALL linear functions are 1-1.



[EX2] Prove $y = x^3$ is 1-1.

If $y_1 = y_2$. then $x_1^3 = x_2^3$.

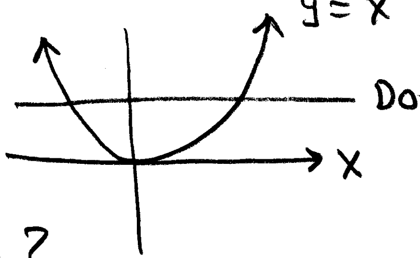
So $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$,

thus $x_1 = x_2$.

□

→

IS $y = x^2$ 1-1?



Does NOT Look 1-1

where does proof fail?

If $y_1 = y_2$. then $x_1^2 = x_2^2$.

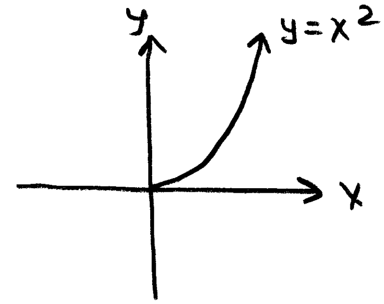
$\Rightarrow \sqrt{x_1^2} = \sqrt{x_2^2}$

$\Rightarrow x_1 = x_2$

BOGUS. $2^2 = (-2)^2$ but $2 \neq -2$.

Sometimes we can restrict the domain so that at least some of the function has an inverse.

$f(x) = x^2$, $x \geq 0$ has an inverse

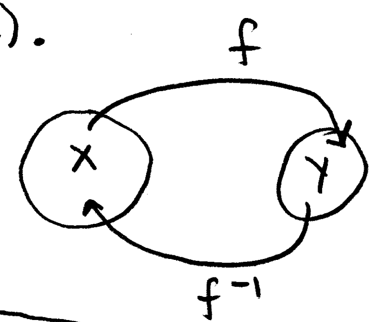


★ This is an important tactic (as you will see in trig)

Proving a function must have an inverse and actually finding the inverse are not the same thing.

Finding inverse

[EX3] If $f(x) = 3x^2 + 2$, find $f^{-1}(x)$.



$\boxed{R_f \text{ is } D_{f^{-1}}}$

"x and y switch roles"

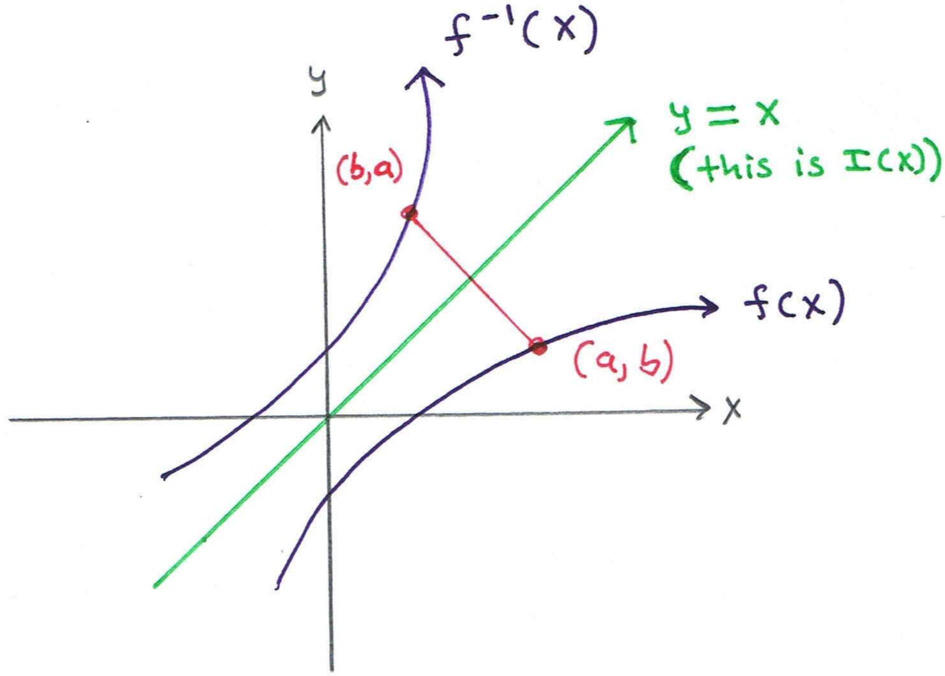
soln ① write $x = 3y + 2$ (x, y switch roles)

② Solve for y.

$$f^{-1}(x) = y = \frac{x-2}{3}$$

means write X as a function of Y.

Graphically



Do you understand why $y = x$ is the line of symmetry for f and f^{-1}?

— Because the components of (a,b) on f must "switch roles" to get in f^{-1}. that makes $y = x$ the line of Symmetry!

— How nice that $y = x$ is also the identity elem. for the set of functions under operation of Composition.

p.s. Under composition, the set of functions is a non-communtative group.